Induction (Again...)

Prop: For all integers $n \geq 4$ we have $2^n \leq n!$.

(**Defn:** For $n \in \mathbb{N}$, $n!$ (i.e. the factorial of n) is defined (recursively) by $0! = 1$ and $n! = (n-1)! \cdot n$ for all $n \geq 1$.)

$\forall_{n \geq 4}\left[2^n \leq n!\right]$

Pf: We proceed by induction on n.

Base Case: For $n = 4$ we have

$2^n = 2^4 = 16$ and $n! = 4! = 4 \cdot 3 \cdot 2 \cdot 1 \cdot 1 = 24$.

Thus $2^n = 16 \leq 24 = n!$, which verifies the base case.

Inductive Step: Suppose $2^n \leq n!$ for some $n \geq 4$. Now

$\forall_{n \geq 4}\left[P(n) \to P(n+1)\right]$

we have

$$2^{n+1} \overset{\text{arithmetic}}{=} 2 \cdot 2^n \overset{\text{IH}}{\leq} 2(n!) \overset{2 \leq 4}{\leq} 4(n!)$$

$$\overset{4 \leq n}{\leq} n(n!) \leq (n+1)\cdot(n!) = (n+1)!$$

($n \leq n+1$; definition of $(n+1)!$)

Hence $2^{n+1} \leq (n+1)!$, and we have shown directly that the inductive step holds.

Hence, by mathematical induction we have $2^n \leq n!$ for all $n \geq 4$.

∎

BAD INDUCTION PROOFS

~~Prop~~: For all $n \geq 0$, we have $n^3 \leq n^2$. FALSE

VERY BAD PROOF...

Pf: We proceed by induction on n. ← Very False.

Base case: When $n = 0$, we have $0^3 = 0 \leq 0 = 0^2$, so the base case holds. still good. ✓

Induction Step: Assume the result holds for some $n \geq 0$. [In particular, we know it holds for $n = 0$.] We now prove the result holds for $n+1$. In particular

$$(n+1)^3 = (0+1)^3 = 1^3 = 1 \leq 1 = 1^2 = (0+1)^2 = (n+1)^2$$

[Hence we've verified the inductive step.]

We conclude the original statement holds by induction ☒

that is the lie! This proof only shows $0 \leq 0$ and $1 \leq 1$

(FALSE) Prop: Everyone on earth has the same name.

(FALSE) Proof: We prove that every set of n people shares a common name, by induction on n.

Base Case: When $n = 1$, we have just 1 person x, so the base case holds by noting everyone in $\{x\}$ has the same name as everyone else. ← This isn't covered by our base cases!

Inductive Step: Suppose every set of $k \geq 2$ people shares a common name, and assume S is a set of $k+1$ people. We may label S as

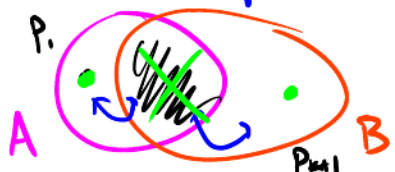
$$S = \{p_1, p_2, \cdots, p_{k+1}\}.$$

Define $A = \{p_1, p_2, \cdots, p_k\}$ and $B = \{p_2, p_3, \cdots, p_{k+1}\}$.

Both A and B have precisely k people. Hence everyone in set A shares the same name, and everyone in set B shares the same name. Now

From A

$p_1, p_2, \cdots, p_k$ all have the same name, x; ~~thus p_1 and p_2 share the same name.~~ Moreover, $p_2, p_3, \cdots, p_{k+1}$ from B all share the same name, y; ~~in particular, p_2 and p_{k+1} have the same name.~~ Now, p_2 has the name x and y; because a person has only one name, $x = y$.



What if $k = 1$? Implicitly assumed $k \geq 2$.

Hence, everyone in $S = A \cup B$ share the same name, and we've verified the inductive step.

Hence the original statement holds by induction. ☑

What we just saw: Both the base case and the inductive step are essential to proof by induction

→

We've been using "weak mathematical induction":

↳ $P(0) \wedge \forall_{k \geq 0} [P(k) \to P(k+1)] \equiv \forall_{n \geq 0} [P(n)]$.

"Strong mathematical induction" also works:

$$P(0) \wedge P(1) \wedge \cdots \wedge P(j) \wedge \forall_{k \geq j} \left[\left(\forall_{0 \leq i \leq k} P(i) \right) \to P(k+1) \right]$$

$$\equiv \forall_{n \geq 0} [P(n)].$$

Fun Fact: Weak induction and Strong induction are both "two sides of the same coin". Both are valid proof methods...